EXPERIMENTAL DESIGN OF MEASUREMENTS FOR THE SOLUTION OF COEFFICIENT-TYPE INVERSE HEAT-CONDUCTION PROBLEMS

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A numerical method is proposed for optimization of the spatial placement of a fixed number of temperature sensors in the solution of coefficient-type inverse heat-conduction problems.

A large class of promising methods for the analysis and interpretation of data from transient thermophysical experiments is based on the solution of coefficient-type inverse heat-conduction problems [1, 2]. In this approach the coefficients (thermophysical characteristics) in the heat-conduction equation are determined from the known boundary and initial conditions and from the data of transient temperature measurements at a finite number of spatially distributed points of the analyzed body. It is assumed that the heat-conduction equation describes the investigated heat-transfer process with sufficient accuracy and that the coefficients of the equation are equivalent in the general case to certain "effective" thermophysical characteristics of the real material.

We consider the quasilinear heat-conduction boundary-value problem with Dirichlet and Neumann boundary conditions:

$$C(T)\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right),\quad 0<x<b,\quad 0<\tau\leqslant\tau_m, \tag{1}$$

$$T(x,\ 0)=\gamma(x),\quad 0\leqslant x\leqslant b, \tag{2}$$

$$T(0,\ \tau)=g_1(\tau)\ \text{ or }\ -\lambda(T(0,\ \tau))\frac{\partial T(0,\ \tau)}{\partial x}=q_1(\tau), \tag{3}$$

$$T(b,\ \tau)=g_2(\tau)\ \text{ or }\ -\lambda(T(b,\ \tau))\frac{\partial T(b,\ \tau)}{\partial x}=q_2(\tau), \tag{4}$$

where $\gamma(x)$, $g_1(\tau)$, $g_2(\tau)$, $q_1(\tau)$, $q_2(\tau)$ are known functions.

Depending on the a priori information about the characteristics C(T) and λ(T), coefficient-type inverse heat-conduction (IHC) problems can be stated in different ways, namely: to determine either of the characteristics or both of them simultaneously. The input data for the solution of IHC problems are provided by measurements of the temperature at one or more points of the investigated sample:

$$T(X_i,\ \tau)=f_i(\tau),\ i=1,\ 2,\ \ldots,\ N. \tag{5}$$

The usual criterion for the selection of the unknown characteristics is the mean-square deviation of the temperature values calculated by means of the mathematical model (1)-(4) at the sensor placement sites from the experimentally measured values:

$$I=\sum_{i=1}^{N}\int_0^{\tau_m}[T(X_i,\ \tau)-\hat{f}_i(\tau)]^2\,d\tau. \tag{6}$$

It must be emphasized that the minimum required number of temperature sensors and the possible region of their placement are completely determined by an analysis of the conditions for the existence and uniqueness of a solution of the corresponding inverse problem. For ex-

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ample, the results of [3] indicate that in order to determine either of the functions C(T) or λ(T) unambiguously from conditions (1)-(6) it is necessary to perform a transient measurement of the temperature at one point. This point must be situated in the interior of the investigated spatial region under specified Dirichlet boundary conditions or it can be situated on one of the boundaries under a Neumann-type boundary condition. However, if the problem of identifying both characteristics is analyzed, a unique solution of such an IHT problem is provided by measuring the temperature at two different points and specifying a Neumann boundary condition on at least one of the boundaries, in which case the heat flux must be nonvanishing. Requirements of this type on the measurements dictate the minimum possible experimental information that is required in principle for the solution of a specific inverse problem. If a greater number of measurements is performed, the problem becomes overdetermined.

Effective iterative regularizing algorithms for the solution of coefficient-type IHT problems can be formulated on the basis of gradient methods for the minimization of the criterion (6) with parametrization of the unknown functions in the form

$$P(T) = \sum_{k=1}^{m} p_k \varphi_k(T), \tag{7}$$

where P(T) is the sought-after characteristic and $\varphi_k(T)$, k = 1, 2, ..., m, is a system of basis functions, in the role of which it is convenient to use cubic B-splines [4].

In this case the inverse problem is reduced to the determination of the vector of parameters $P = \{p_k, k = 1, 2, \ldots, M\}$, the composition of which includes the coefficients of the parametric representation of all the characteristics to be determined in the problem. The most efficient way to compute the components of the gradient of the functional (6) from the unknown parameters is to use the solution of the dual boundary-value problem corresponding to the primary problem. Algorithms of the given type have been proposed in [5-8]. Some results pertaining to the substantiation of the regularizing properties of the interative algorithms are presented in [8, 9].

Mathematical modeling data [10] show that the error of determination of the thermophysical characteristics from the solution of the IHT problem may depend significantly on the spatial placement of the temperature sensors in the investigated body. The proper setup of the thermophysical experiment requires the solution of an optimum experimental design problem, viz.; to place a fixed number of sensors in the sample in such a way as to minimize the error of identification of the required characteristics. The stated problem can be solved on the basis of the fundamental principles of experimental design theory for distributed-parameter systems (see, e.g., [11, 12]).

We consider the experimental design problem for measurements performed in order to determine the two characteristics C(T) and λ(T) from conditions (1)-(6) in the presence of excess experimental information (N > 2). This approach can be used to analyze other important practical situations as special cases.

Using the parametrization of the sought-after functions

$$C(T) = \sum_{k=1}^{m_1} C_k \varphi_{1k}(T), \quad \lambda(T) = \sum_{k=m_1+1}^{m_1+m_2} \lambda_k \varphi_{2k}(T), \tag{8}$$

we form the vector of unknown parameters $\mathbf{P} = \{p_k, k = 1, 2, \ldots, M\}$, where $p_k = C_k$, $k = 1, 2, \ldots, m_1$; $p_k = \lambda_k$, $k = m_1 + 1, m_1 + 2, \ldots, M = m_1 + m_2$.

Next we introduce a measurement design in the form of a vector of space coordinates of the placement points of the temperature sensors:

$$\mathbf{X} = \{X_i, i = 1, 2, \ldots, N\}. \tag{9}$$

The error of determination of the vector P from the solution of the inverse problem (1)-(6) is related continuously to the normalized Fisher information matrix [11, 12]

$$F(\mathbf{X}) = \frac{1}{N} \{\Phi_{kj}, k, j = 1, 2, \ldots, M\}, \tag{10}$$

where

$$\Phi_{kj} = \sum_{i=1}^{N} \int_{0}^{\tau_m} \theta_k(X_i, \tau)\,\theta_j(X_i, \tau)\,d\tau;$$

$\theta_k(X_i, \tau)$ are the sensitivity function computed at the sensor placement points: $\theta_k(X_i, \tau) = \partial T(X_i, \tau)/\partial p_k$, $k = 1, 2, \ldots, M$.

To obtain the relations governing the sensitivity functions we differentiate Eqs. (1)-(4) m_1 times with respect to C_k, $k = 1, 2, \ldots, m_1$, and m_2 times with respect to λ_k, $k = m_1 + 1, m_1 + 2, \ldots, M$. Then, making use of Eq. (8), after suitable transformations we obtain

$$C(T)\frac{\partial\theta_k}{\partial\tau} = \frac{\partial}{\partial x}(\lambda(T))\frac{\partial\theta_k}{\partial x} + \frac{\partial T}{\partial x}\frac{\partial\lambda}{\partial T}\frac{\partial\theta_k}{\partial x} + \left[\frac{\partial^2 T}{\partial x^2}\frac{\partial\lambda}{\partial T} + \left(\frac{\partial T}{\partial x}\right)^2\frac{\partial^2\lambda}{\partial T^2} - \frac{\partial T}{\partial\tau}\frac{\partial C}{\partial T}\right]\theta_k + S_k, \quad (11)$$

$$0 < x < b, \quad 0 < \tau \leqslant \tau_m,$$
$$\theta_k(x, 0) = 0, \quad 0 \leqslant x \leqslant b, \quad (12)$$

$$\theta_k(0, \tau) = 0 \quad \text{or} \quad \lambda(T(0, \tau))\frac{\partial\theta_k(0, \tau)}{\partial x} + Q_k(0, \tau)\,\theta_k(0, \tau) + R_k(0, \tau) = 0, \quad (13)$$

$$\theta_k(b, \tau) = 0 \quad \text{or} \quad \lambda(T(b, \tau))\frac{\partial\theta_k(b, \tau)}{\partial x} + Q_k(b, \tau)\,\theta_k(b, \tau) + R_k(b, \tau) = 0, \quad (14)$$

where

$$S_k = \begin{cases} -\dfrac{\partial T}{\partial\tau}\varphi_{1k}(T), & k = 1, 2, \ldots, m_1, \\ \dfrac{\partial^2 T}{\partial x^2}\varphi_{2k}(T) + \left(\dfrac{\partial T}{\partial x}\right)^2\dfrac{d\varphi_{2k}}{dT}, & k = m_1 + 1, m_1 + 2, \ldots, M; \end{cases}$$

$$Q_k = \begin{cases} 0, & k = 1, 2, \ldots, m_1, \\ \dfrac{\partial T}{\partial x}\dfrac{\partial\lambda}{\partial T}, & k = m_1 + 1, m_1 + 2, \ldots, M; \end{cases}$$

$$R_k = \begin{cases} 0, & k = 1, 2, \ldots, m_1, \\ \dfrac{\partial T}{\partial x}\varphi_{2k}(T), & k = m_1 + 1, m_1 + 2, \ldots, M. \end{cases}$$

To contrast with the primary problem (1)-(4), the boundary-value problems for the sensitivity functions are linear. For their solution it is required to know the temperature field $T(x, \tau)$ in the investigated sample.

The information matrix (10) characterizes the total sensitivity of the analyzed system in the entire set of measurement points to the variation all components of the vector of unknown parameters. The given optimum measurement design problem entails finding a design $\mathbf{X}$ for which the total sensitivity of the system in the adopted sense will be a maximum.

Various criteria are used for the optimization of the experimental conditions. The so-called D-optimum design [11, 12] is widely used to ensure the minimum error of estimation of the unknown parameters. In this case the measurement design can be determined from the condition of the maximum of the determinant of the normalized information matrix:

$$D^* = \max_{\mathbf{X}}[\det F(\mathbf{X})]. \quad (15)$$

We note that the components of the vector $\mathbf{X}$ in the given problem must satisfy the constraints

$$0 < X_i < b, \quad i = 1, 2, \ldots, N, \quad (16)$$

when Dirichlet-type boundary conditions are specified in the IHT problem, and

$$0 \leqslant X_i \leqslant b, \quad i = 1, 2, \ldots, N, \quad (17)$$

when Neumann-type boundary conditions are known.

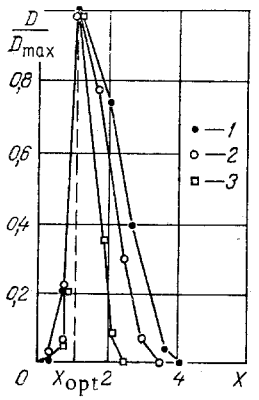

Fig. 1. Optimum placement (X, mm) of a temperature sensor for the determination of the thermal conductivity coefficient. 1) M = 3; 2) 4; 3) 5.

It must be emphasized that the elements of the normalized information matrix F(X) [see relations (11)-(14)] and, hence, the measurement design X depend on the vector of unknown parameters P. This situation is attributable to the strongly nonlinear dependence of the temperature on the unknown parameters and is typical of measurement designs for the solution of inverse problems in mathematical physics. Accordingly, it is only meaningful to speak of locally optimum designs, which are formulated with the use of a priori information about the unknown parameters [11].

The above-described measurement-design method is implemented in the form of a computational algorithm and computer program. The corresponding boundary-value problems are solved numerically with the application of a monotonic approximation scheme [13]. The system of difference equations is solved by the double-sweep (modified Gaussian elimination) method. In the nonlinear case, iterations are carried otu with respect to the coefficients. After the elements of the normalized information matrix have been calculated, the conditional optimization problem (15), (16) or (15), (17) is solved. The following reasonably simple computational algorithm of direct sequential search for an optimum measurement design on a fixed differencing grid with respect to the space coordinate is used here. All N temperature sensors are placed at the first node. Then the N-th sensor "runs through" all nodes of the grid, and the value of det F(X) is computed at each of them. Next, the (N − 1)-st and N-th sensors are placed at the second node, and the others are left at the first node. The N-th sensor again "runs through" all nodes of the grid, but now beginning with the second node. The process is continued until the two sensors indexed N and N − 1 are located at the last node. After this, the (N − 2)-nd, (N − 1)-st, and N-th sensors are placed at the second node of the grid, and the computations are repeated. The sequential search process is terminated when all N sensors are located at the last node of the differencing grid. The proposed algorithm can be used to compute the optimum placement of sensors within error limits corresponding to half the step of the space grid.

Methodological control examples have been calculated with the application of the method developed here. Some results are given in Fig. 1.

The following mathematical modeling procedure is used. The inverse problem of determining the thermal conductivity coefficient under Dirichlet boundary conditions is solved for a sample in the form of an unbounded plate with a thickness of 8 mm. The function $\lambda(T) = 0.5\,[1 + (T/1000)^2]$ is specified a priori. The temperature field is calculated with Neumann boundary conditions for values of the heat flux at the boundaries $g_1(\tau) = 1.5 \cdot 10^5$ and $g_2(\tau) = 0$. The remaining initial data are taken equal to $C(T) = 4.0 \cdot 10^6$, $\tau_m = 60$, $\gamma(x) = 0$. It is assumed here that the readings of a single temperature sensor are used in solving the IHT problem, and the number M of zones in the spline approximation of the function $\lambda(T)$ is varied.

The results demonstrate the existence of a fairly limited spatial region in which the temperature sensors must be placed. This region is further restricted with an increase in the number M.

It is important to note that the possibility of formulating only locally optimum measurement designs in coefficient-type IHT problems indicates the iterative nature of the experimental design and characteristic-identification process. The general sequence of operations must be as follows. An initial approximation of the unknown vector P is specified on the basis of a priori information. Next, the sufficient number of temperature sensors for the unique solution of the inverse problem is determined, along with their optimum placement with respect to the space coordinate. Then the real experiment and measurements must be carried out. The experimental data are processed by the methods of coefficient-type IHT problems, and a new approximation of the vector of unknown parameters is determined. This approximation is used as the initial (a priori) information for the next iteration in the general characteristic-identification procedure. The successive-approximation process must be continued until the results in two successive iterations coincide within prescribed error limits.

We note that the rate of convergence of the iterative identification process described here can be increased by means of a more detailed experimental design. The detailed refinement consists in analyzing the a priori information about the unknown parameters in the form of a certain set of values: $\mathbf{P}_{\min} \leqslant \mathbf{P} \leqslant \mathbf{P}_{\max}$. A certain average measurement design can be obtained on the basis of such an analysis and used in the next iteration.

NOTATION

T, temperature; x, space coordinate; τ, time; C(T), volume specific heat; $\lambda(T)$, thermal conductivity; b, sample thickness; $\gamma(x)$, initial temperature distribution; $g_1(\tau)$, $q_1(\tau)$, $g_2(\tau)$, $q_2(\tau)$, thermal regime at boundaries; X, coordinate of temperature sensor placement; N, number of sensors; $f(\tau)$, experimentally measured temperatures; $\theta(x, \tau)$, sensitivity function; F(X), normalized Fisher information matrix; $\mathbf{P} = \{p_k, k = 1, 2, \ldots, M\}$, vector of parameters to be identified; $\mathbf{X} = \{X_i, i = 1, 2, \ldots, N\}$, measurement design.

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